Prepotential formulation of SU(3) lattice gauge theory

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Abstract

The SU(3) lattice gauge theory is reformulated in terms of SU(3) prepotential harmonic oscillators. This reformulation has enlarged $SU(3) \otimes U(1) \otimes U(1)$ gauge invariance under which the prepotential operators transform like matter fields. The Hilbert space of SU(3) lattice gauge theory is shown to be equivalent to the Hilbert space of the prepotential formulation satisfying certain color invariant Sp(2,R) constraints. The SU(3) irreducible prepotential operators which solve these Sp(2,R) constraints are used to construct SU(3) gauge invariant Hilbert spaces at every lattice site in terms of SU(3) gauge invariant vertex operators. The electric fields and the link operators are reconstructed in terms of these SU(3) irreducible prepotential operators. We show that all the SU(3) Mandelstam constraints become local and take very simple form within this approach. We also discuss the construction of all possible linearly independent SU(3) loop states which solve the Mandelstam constraints. The techniques can be easily generalized to SU(N).

1 Introduction

The reformulation of gauge theories in terms of gauge invariant Wilson loops and strings carrying fluxes of the corresponding gauge group is an old problem in quantum field theory [1, 2, 3, 4, 5, 6]. The motivation to go from colored gluons and quarks to colorless loops and string degrees of freedom comes from the expectation that the latter framework is better suited to analyze and understand long distance non-perturbative issues like color confinement in QCD. Infact, the lattice formulation of gauge

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theories was a step in this direction where one directly works with link operators (instead of gauge connections) which create and destroy abelian or non-abelian loop fluxes on lattice links. However, the two major obstacles in this loop, string approach to QCD are the non-locality and proliferation of loops and string states [7]. The non-locality is obvious as the loops and strings can be of any shapes and sizes. The problem of proliferation exists because the set of all Wilson loop states forms a highly over complete basis. This is because not all loop states are mutually independent (see section 3.3 and 4.6). Their relationships are expressed by the Mandelstam constraints. The Mandelstam constraints, in turn, are difficult to solve because of their non-locality (section 3.3 and 4.6). *Therefore, it is important to explore new descriptions of QCD where the loop, string states and their dynamics as well as the associated Mandelstam constraints can be analyzed locally.* As shown in [8, 9], the prepotential approach to lattice gauge theories provides such a platform. More precisely, this approach allows us to analyze and solve the Mandelstam constraints locally at each lattice site without all the irrelevant non-local details associated with the loop states (section 3.3 and 4.6). Towards this goal, a complete analysis was carried out for SU(2) lattice gauge theory and all mutually independent loop states were constructed in terms of prepotential operators in [8, 9]. The purpose and motivation of this work is to analyze lattice QCD or SU(3) lattice gauge theory within the prepotential framework. As we will see, there are many new issues which come up due to very different flux properties of SU(3) and SU(2) lattice gauge theories.

The prepotential operators are harmonic oscillators belonging to the fundamental representations of the gauge group. Further, unlike link operators which create and destroy fluxes on the links, the prepotential operators are associated with the sites and create or destroy smallest units of group fluxes at the corresponding lattice sites. In the case of SU(2) lattice gauge theory [8], the prepotential approach enabled us to cast all the SU(2) Mandelstam constraints in their local form. Further, all possible mutually orthonormal loop states were explicitly constructed in terms of the prepotential operators. The dynamics of these orthonormal SU(2) loop states was shown to be governed by 3-nj Wigner coefficients. Infact, similar results have been obtained in the context of duality transformations in SU(2) lattice gauge theories in [10, 11, 12, 13, 14]. More precisely, the SU(2) gauge invariant basis labeled by (dual) angular momentum quantum numbers, describing two dimensional triangulated surfaces, in [11] is exactly same as the SU(2) loop basis in [8] labeled by "linking quantum numbers" which describe one dimensional loops. In [6, 15, 16, 17] different computational schemes to identify independent SU(2) loops were proposed. In [6, 15] loop Hamiltonians are computed in the above schemes retaining small loops carrying small fluxes[1]. In the context of loop quantum gravity, SU(2) spin networks carrying SU(2) fluxes which describe geometry of space time have been extensively studied [18, 19]. The SU(2) Schwinger boson or equivalently prepotential techniques studied in [8, 9] can also be naturally applied to study the spin networks in loop quantum gravity as the fluxes in the spin networks are created by Schwinger bosons. This approach leads to many technical simplifications in the construction of spin networks and has been discussed extensively in [20]. On the other hand, in the context of QCD with SU(3) gauge group hardly any work has been done in these directions. In particular, it is important to construct and analyze all independent SU(3) loop states ("SU(3) spin

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[1]Note that the prepotential formulation resolve the issues of over completeness of SU(2) loop states and their dynamics exactly without any assumptions.

networks") and study their dynamics. In the context of QCD, this analysis will be useful to analyze the spectrum of QCD Hamiltonian in terms of loops near the continuum limit where large loops carrying large fluxes are expected to dominate. The exact minimal loop basis containing arbitrarily large loops with all possible fluxes will allow us to analyze the spectrum without any spurious loop degrees of freedom.

In this work we show that the SU(3) lattice gauge theory can also be completely described in terms of SU(3) irreducible prepotentials with $SU(3) \otimes U(1) \otimes U(1)$ gauge invariance. Under $SU(3) \otimes U(1) \otimes U(1)$ gauge transformations the prepotentials transform like charged matter fields. All the non-local SU(3) Mandelstam constraints in term of the link operators are cast into their local forms with the help of SU(3) gauge invariant prepotential vertex operators which are defined at lattice sites (section 3.3 and 4.6). We briefly discuss how to get all the solutions of SU(3) Mandelstam constraints in the form of all possible independent SU(3) loop states.

The paper is organized as follows. In Section 2, we briefly discuss the Hamiltonian formulation of SU(N) lattice gauge theory. This section sets up the notations and makes the paper self contained. The section 3 starts with a brief summary of the SU(2) prepotential approach to lattice gauge theory [8, 9]. This overview illustrates all the essential ideas involved in simplifying the Mandelstam constraints and getting all their solutions in the simpler SU(2) case before dealing with their more involved SU(3) analogues. In addition, this section also helps us to highlight some completely new issues and difficulties one confronts on going from SU(2) to SU(3) lattice gauge theory. Section 4 discusses SU(3) lattice gauge theory in terms of prepotential operators. In sections 4.1 and 4.2 we study and classify the SU(3) prepotential Hilbert space $\mathcal{H}_p$ according to SU(3) invariant Sp(2,R) quantum numbers [21]. In section 4.3, we show that the Hilbert space of SU(3) gauge theory $\mathcal{H}_g$ is a tiny subspace of $\mathcal{H}_p$ which satisfies certain Sp(2,R) constraints. Section 4.4 deals with SU(3) irreducible prepotential operators [22] which are solutions of the above Sp(2,R) constraints and therefore directly create the gauge theory Hilbert space $\mathcal{H}_g$. The explicit construction of $SU(3)$ link operators and electric fields in terms of the SU(3) irreducible prepotentials is given in section 4.5. In section 4.6, with the help of SU(3) irreducible prepotential operators, we construct all possible SU(3) gauge invariant vertices at a given lattice site which in turn cast all SU(3) Mandelstam constraints in their local forms. Having made them local, section 4.6.1 discusses how to solve these infinite sets of constraints at every lattice site exactly. We then briefly discuss the prepotential formulation of SU(N) lattice gauge theory. We end the paper with a brief summary and discussion on related issues.

2 SU(N) Hamiltonian formulation

The Hamiltonian of $SU(N)$ lattice gauge theory is:

$$H = \sum_{n,i} \sum_{a=1}^{N^2-1} E^a(n,i) E^a(n,i) + K \sum_{\text{plaquette}} Tr \left(U_{\text{plaquette}} + U^\dagger_{\text{plaquette}} \right) \tag{1}$$

with,

$$U_{\text{plaquette}} = U(n,i) U(n+i,j) U^\dagger(n+j,i) U^\dagger(n,j),$$

where K is the coupling constant, a$(= 1, 2, \cdots, (N^2 - 1))$ is the color index. In (1) the kinematical operators E and U can be understood as follows. Each link (n,i) is associated with a SU(N) symmetric top, whose configuration (i.e the rotation matrix from space fixed to body fixed frame) is given by the operator valued $(N \times N)$ SU(N) matrix U(n,i). Let $E_L^{\mathrm{a}}(n, i), E_R^{\mathrm{a}}(n + i, i)$ denote the conjugate left and right electric fields with the quantization rules [5]:

$$[E_L^{\mathrm{a}}(n, i), U^{\alpha}{}_{\beta}(n, i)] = -(T^{\mathrm{a}} U(n, i))^{\alpha}{}_{\beta}, \qquad [E_R^{\mathrm{a}}(n + i, i), U_{\beta}^{\alpha}(n, i)] = (U(n, i) T^{\mathrm{a}})^{\alpha}{}_{\beta}. \qquad (2)$$

In (2), T^{a} are the generators in the fundamental representation of $SU(N)$ and satisfy: $[T^{\mathrm{a}}, T^{\mathrm{b}}] = if^{\mathrm{abc}}T_{\mathrm{c}}$ where f^{abc} are the SU(N) structure constants. The quantization rules (2) clearly show that $E_L(n, i)$ and $E_R(n + i, i)$ are the generators of left and the right gauge transformations in (7). Infact, the right generators $E_R^{\mathrm{a}}(n + i, i)$ are the parallel transport of the left generator $E_L^{\mathrm{a}}(n, i)$ on the link (n, i):

$$E_R(n + i, i) = -U^{\dagger}(n, i) E_L(n, i) U(n, i). \qquad (3)$$

In (3), $E_R(n + i, i) \equiv \sum_{\mathrm{a}} E_R^{\mathrm{a}}(n + i, i) T^{\mathrm{a}}$ and $E_L(n, i) \equiv \sum_{\mathrm{a}} E_L^{\mathrm{a}}(n, i) T^{\mathrm{a}}$. The left and the right electric fields on every link, being the SU(N) rotation generators, satisfy:

$$[E_L^{\mathrm{a}}(n, i), E_L^{\mathrm{b}}(n, i)] = if_{\mathrm{abc}} E_L^{\mathrm{c}}(n, i), \quad [E_R^{\mathrm{a}}(n, i), E_R^{\mathrm{b}}(n, i)] = if_{\mathrm{abc}} E_R^{\mathrm{c}}(n, i). \qquad (4)$$

Further, using (3), it is easy to show that E_L^{a} and E_R^{a} commute amongst themselves:

$$\left[E_L^{\mathrm{a}}(n, i), E_R^{\mathrm{b}}(m, j)\right] = 0. \qquad (5)$$

and therefore mutually independent. By construction on each link they always satisfy the constraints:

$$\sum_{\mathrm{a}=1}^{N^2-1} E^{\mathrm{a}}(n, i) E^{\mathrm{a}}(n, i) \equiv \sum_{\mathrm{a}=1}^{N^2-1} E_L^{\mathrm{a}}(n, i) E_L^{\mathrm{a}}(n, i) = \sum_{\mathrm{a}=1}^{N^2-1} E_R^{\mathrm{a}}(n + i, i) E_R^{\mathrm{a}}(n + i, i). \qquad (6)$$

The Hamiltonian in (1) involves the squares of either left or the right electric fields. Under gauge transformation the left electric field and the link operator transforms as:

$$U(n, i) \rightarrow \Lambda(n) U(n, i) \Lambda^{\dagger}(n + i),$$
$$E_L(n, i) \rightarrow \Lambda(n) E_L(n, i) \Lambda^{\dagger}(n), \quad E_R(n + i, i) \rightarrow \Lambda(n + i) E_R(n + i, i) \Lambda^{\dagger}(n + i). \qquad (7)$$

The Hamiltonian (1) and the basic commutation relations (2) are invariant under the SU(N) gauge transformations (7). From (7), the $SU(N)$ Gauss law constraint at every lattice site n is

$$G(n) = \sum_{i=1}^{d} \left(E_L^{\mathrm{a}}(n, i) + E_R^{\mathrm{a}}(n, i) \right) = 0, \forall n. \qquad (8)$$

It is convenient to define the left and right strong coupling vacuum state $|0\rangle_L$ and $|0\rangle_R$ on every link which are annihilated by their corresponding electric fields:

$$E_L^{\mathrm{a}}(n, i)|0, (n, i)\rangle_L = 0, \qquad E_R^{\mathrm{a}}(n + i, i)|0, (n + i, i)\rangle_R = 0, \quad \forall \text{ links } (n, i). \qquad (9)$$

We will denote the vacuum state on a link by $|0\rangle \equiv |0, (n, i)\rangle_L \otimes |0, (n, i)\rangle_R$, suppressing all the link as well as L, R indices. The quantization rules (2) show that the link operators $U^{\alpha}{}_{\beta}(n, i)$ acting on the strong coupling vacuum (9) create SU(N) fluxes on the links. As an example, using (2):

$$E_L^2(n, i)\left(U^{\alpha}{}_{\beta}|0\rangle\right) = E_R^2(n + i, i)\left(U^{\alpha}{}_{\beta}|0\rangle\right) = \frac{1}{2N}\left(N^2 - 1\right)\left(U^{\alpha}{}_{\beta}|0\rangle\right). \qquad (10)$$

The higher SU(3) irreducible flux eigenstates of E_L^2 and E_R^2 on a link can be obtained by considering the states $U^{\alpha_1}{}_{\beta_1} U^{\alpha_2}{}_{\beta_2} \cdots U^{\alpha_1}{}_{\beta_1}|0\rangle$ and symmetrizing the α and therefore also β indices according to certain SU(N) Young tableau. We will discuss this issue again in section 3 and section 4 in the specific context of SU(2) and SU(3) groups.

3   Prepotentials in SU(2) lattice gauge theory

In this section we define SU(2) prepotential operators. Using the Schwinger bosons construction of the angular momentum algebra (4), the left and the right electric fields on a link (n, i) can be written as:

$$\text{Left electric fields:} \qquad E_L^{\mathrm{a}}(n, i) \;\equiv\; a^\dagger(n, i; L)\frac{\sigma^{\mathrm{a}}}{2}a(n, i; L), \qquad (11)$$

$$\text{Right electric fields:} \quad E_R^{\mathrm{a}}(n + i, i) \;\equiv\; a^\dagger(n + i, i; R)\frac{\sigma^{\mathrm{a}}}{2}a(n + i, i; R).$$

In (11), $a_\alpha(n, i; l)$ and $a_\alpha^\dagger(n, i; l)$ are the doublets of harmonic oscillator creation and annihilation operators with $l = L, R, \alpha = 1, 2$. We have used Schwinger boson construction [23] of angular momentum algebra in (11). Like $E_L^{\mathrm{a}}(n, i)$ and $E_R^{\mathrm{a}}(n + i, i)$, the locations of $a(n, i, L), a^\dagger(n, i, L)$ and $a(n + i, i, R), a^\dagger(n + i, i, R)$ are on the left and the right of the link (n, i). For notational convenience we suppress the link indices and denote $a^\dagger(n, i, L)$ and $a^\dagger(n + i, i, R)$ by $a^\dagger(L)$ and $a^\dagger(R)$ respectively. This is clearly illustrated in Figure 1. The link indices will be explicitly shown whenever we work with more than one link. Note that the relations (11) imply that the strong coupling vacuum (9) is the harmonic oscillator vacuum. Substituting the electric fields (11) in terms of Schwinger bosons in the electric field constraints (6), we get $a^\dagger(n, i; L) \cdot a(n, i; L) = a^\dagger(n + i, i; R) \cdot a(n + i, i; R)$. We will come back to this issue again in section 3.1.
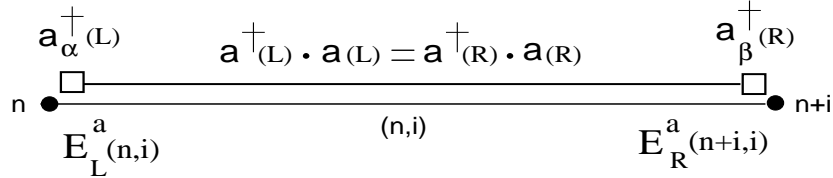

Figure 1: The left and right electric fields and the corresponding prepotentials in SU(2) lattice gauge theory. We have denoted $a^\dagger(n, i, L)$ and $a^\dagger(n+i, i, R)$ by $a^\dagger(L)$ and $a^\dagger(R)$ respectively. The unoriented abelian flux line connecting them represents the U(1) Gauss law (18) constraint.

Under SU(2) gauge transformation with the generator G(n) in (8), the prepotential harmonic oscillator transform as SU(2) doublets[2]:

$$a_\alpha^\dagger(L) \to a_\beta^\dagger(L) \left(\Lambda_L^\dagger\right)^\beta{}_\alpha, \qquad a_\alpha^\dagger(R) \to a_\beta^\dagger(R) \left(\Lambda_R^\dagger\right)^\beta{}_\alpha$$

$$a^\alpha(L) \to \left(\Lambda_L\right)^\alpha{}_\beta \, a^\beta(L), \qquad a^\alpha(R) \to \left(\Lambda_R\right)^\alpha{}_\beta \, a^\beta(R). \qquad (12)$$

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[2]Here we specify the notations in (12): $a_\alpha^\dagger(L) \equiv a_\alpha^\dagger(n, i; L), a_\alpha^\dagger(R) \equiv a_\alpha^\dagger(n + i, i; R)$ are located at the left and right side of the link (n, i) and $\Lambda_L \equiv \Lambda(n)$, $\Lambda_R \equiv \Lambda(n + i)$ as shown in Figure 1 explicitly.

One can also define $\tilde{a}^{\dagger\alpha} = \epsilon^{\alpha\beta}a^\dagger_\beta$ and $\tilde{a}_\alpha = \epsilon_{\alpha\beta}a^\beta$ which under $SU(2)$ transformation transform as a^α and $a^\dagger_\alpha$ respectively. In terms of link operators the basic SU(2) flux states on links can be constructed using the link operators:

$$|j(n,i), m_L(n,i), m_R(n,i)\rangle = \left(U^{\alpha_1}{}_{\beta_1} U^{\alpha_2}{}_{\beta_2} \cdots U^{\alpha_{2j}}{}_{\beta_{2j}} + \cdots (2j)! \text{ permutations} \right)|0\rangle. \qquad (13)$$

In (13), $j_L(n,i) = j_R(n+i,i) \equiv j(n,i)$ because of (6), $m_L = \sum_{i=1}^{2j} \alpha_i$ and $m_R = \sum_{i=1}^{2j} \beta_i$ with $\alpha_i, \beta_i = \pm\frac{1}{2}$. The $(2j)!$ terms in (13) are required to implement the symmetries of SU(2) Young tableau in the left $(\alpha_1 \alpha_2 \cdots \alpha_{2j})$ as well as the right $(\beta_1 \beta_2 \cdots \beta_{2j})$ indices. The gauge theory Hilbert space $\mathcal{H}_g$ is spanned by direct product of states of type (13) on all the lattice links. Note that as the flux value $j \to \infty$ on various links[3], the construction of the gauge theory Hilbert space $\mathcal{H}_g$ through (13) becomes more and more tedious. The basic link states in (13) can be now be disentangled into it's left and right part as:

$$|j(n,i), m_L(n,i), m_R(n,i)\rangle = |j(n,i), m_L(n,i)\rangle_L \otimes |j(n,i), m_R(n,i)\rangle_R, \qquad (14)$$

where,

$$
\begin{aligned}
|j(n,i), m_L(n,i)\rangle_L &= a^\dagger_{\alpha_1}(L) a^\dagger_{\alpha_2}(L) \cdots a^\dagger_{\alpha_n}(L)|0\rangle_L \equiv \hat{\mathcal{L}}_{\alpha_1 \alpha_2 \cdots \alpha_n}|0\rangle_L, \\
|j(n,i), m_R(n,i)\rangle_R &= a^\dagger_{\beta_1}(R) a^\dagger_{\beta_2}(R) \cdots a^\dagger_{\beta_n}(R)|0\rangle_R \equiv \hat{\mathcal{R}}_{\beta_1 \beta_2 \cdots \beta_n}|0\rangle_R.
\end{aligned}
\qquad (15)
$$

In (15), $n = 2j$, $m_L = \sum_{i=1}^{2j} \alpha_i$ and $m_R = \sum_{i=1}^{2j} \beta_i$ with $\alpha_i, \beta_i = \pm\frac{1}{2}$. The operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ are the $SU(2) \otimes U(1)$ flux creation operators at the left and right end of every link. Note that these operators are SU(2) irreducible as they are symmetric in all the SU(2) spin half indices and are defined for later convenience (see section 4.2). From (13) and (15) we conclude that the Hilbert space $\mathcal{H}_p$ created using the prepotential operators on all lattice links is also the SU(2) gauge theory Hilbert space:

$$\mathcal{H}_g \equiv \mathcal{H}_p. \qquad (16)$$

However, the construction of $\mathcal{H}_g$ using the prepotentials (15) is much simpler than the equivalent equivalent construction (13) using the link operators. This simplicity occurs because unlike the link operators $U_{\alpha\beta}(n,i)$ which are associated with links, the prepotential operators are attached to the sites (i.e, left or right ends of every link). Further, all the SU(2) prepotential creation operators commute amongst themselves and we do not need $(2j)!$ terms (as in (13)) to get the symmetries of SU(2) Young tableau. In words, the symmetries of SU(2) Young tableau are inbuilt in SU(2) prepotential operators. We will come back to this symmetry issue (end of section 4.3) and the identification of $\mathcal{H}_g$ with $\mathcal{H}_p$ (16) (see eqns. (35) and (46)) again when we discuss SU(3) lattice gauge theory in terms of prepotential operators.

3.1 $U(1)$ gauge invariance

The defining equations for the prepotential operators are invariant under $U(1) \otimes U(1)$ gauge transformations on every link:

$$a^\dagger_\alpha(L) \to e^{i\theta(L)} a^\dagger_\alpha(L), \qquad\qquad a^\dagger_\alpha(R) \to e^{-i\theta(R)} a^\dagger_\alpha(R). \qquad (17)$$

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[3]These large j configurations are expected to dominate in the continuum $(g \to 0)$ limit.

Note that the above abelian gauge transformations are defined on the two sides of every link and are independent of the SU(2) gauge transformations (12) which are defined at every lattice site. Using (11), the electric field constraints (6) on the links become the number operator constraints in terms of the prepotential operators:

$$\hat{N}(L) \equiv a^\dagger(L) \cdot a(L) = \hat{N}(R) \equiv a^\dagger(R) \cdot a(R) \equiv \hat{N} \tag{18}$$

In (18), $\hat{N} \equiv \hat{N}(n,i)$ and imply $\theta(L) = \theta(R)$ on every link and reduces the extra $U(1) \otimes U(1)$ gauge invariance to U(1). Thus in the prepotential formulation non-abelian fluxes can be absorbed locally at a site and the abelian fluxes spread along the links. Both the gauge symmetries together lead to non-local (involving at least a plaquette) Wilson loop states (see section 3.3).

3.2   SU(2) link operators

The equations (11) already defines the left and right electric fields in terms of the prepotentials. To establish complete equivalence, we now write down the link operators explicitly in terms of the prepotentials. From $SU(2)$ gauge transformations of the link operator in (7) and $SU(2) \otimes U(1)$ gauge transformations (12), (17) of the prepotentials,

$$U^\alpha{}_\beta = \tilde{a}^{\dagger\alpha}(L)\, \eta \, a^\dagger_\beta(R) + a^\alpha(L)\, \theta \, \tilde{a}_\beta(R), \tag{19}$$

where η and θ are functions of $SU(2)$ invariant number operator. The operators $\tilde{a}^{\dagger\alpha}$ and $\tilde{a}_\beta$ are defined after equation (12). The eqn. (19) is graphically illustrated in terms of SU(2) Young tableaues in Figure 2.

Figure 2: The Young tableau interpretation of the SU(2) link operator U in terms of the prepotential operators (19) acting on a state with $n_L = n_R = 2j$. The two terms in (19) correspond to the two sets of Young tableaues on the right hand side of this figure respectively.

In the explicit matrix form the link operator can be written as the product of the left part U_L and the right part U_R as:

$$U = \underbrace{\begin{pmatrix} a^\dagger_2(L)\eta_L & a_1(L)\theta_L \\ -a^\dagger_1(L)\eta_L & a_2(L)\theta_L \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} \eta_R a^\dagger_1(R) & \eta_R a^\dagger_2(R) \\ \theta_R a_2(R) & \theta_R(-a_1(R)) \end{pmatrix}}_{U_R} \tag{20}$$
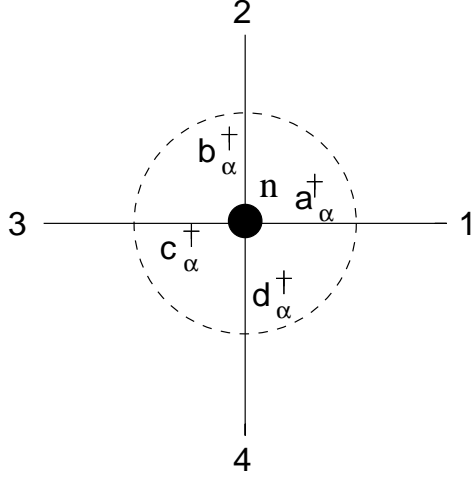
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Figure 3: SU(2) prepotentials associated with a lattice site n in a d = 2 lattice. A SU(2) gauge transformation at site n affects only these prepotentials enabling us to construct SU(2) gauge invariant Hilbert spaces locally at each lattice site.

Where, $\eta_L, \eta_R, \theta_L, \theta_R$ are the left and right invariants constructed out of number operators. From (19) it follows that, $\eta = \eta_L \eta_R$, $\theta = \theta_L \theta_R$. From (20):

$$
U_L^\dagger U_L = \begin{pmatrix} \bar{\eta}_L \left[a^\dagger(L) \cdot a(L) + 2 \right] \eta_L & 0 \\ 0 & \bar{\theta}_L \left[a^\dagger(L) \cdot a(L) \right] \theta_L \end{pmatrix},
$$
$$
U_R U_R^\dagger = \begin{pmatrix} \eta_R \left[a^\dagger(R) \cdot a(R) \right] \bar{\eta}_R & 0 \\ 0 & \theta_R \left[a^\dagger(R) \cdot a(R) + 2 \right] \bar{\theta}_R \end{pmatrix} \tag{21}
$$

Therefore, for $U_{\alpha\beta}$ to be unitary we get:

$$
\eta_L = \frac{1}{\sqrt{a^\dagger(L) \cdot a(L) + 2}}, \; \theta_L = \frac{1}{\sqrt{a^\dagger(L) \cdot a(L)}}, \;\; \eta_R = \frac{1}{\sqrt{a^\dagger(R) \cdot a(R)}}, \; \theta_R = \frac{1}{\sqrt{a^\dagger(R) \cdot a(R) + 2}}. \tag{22}
$$

Note that the operator η_R above is always well defined as it always appears with $a_\alpha^\dagger(R)$ on it's right in (20). The operator θ_L is well defined in (20) as the link operator $U \equiv U_L U_R$ acts on the Hilbert space satisfying the constraints (18). Finally, using $a^\dagger(L) \cdot a(L) = a^\dagger(R) \cdot a(R) \equiv \hat{N}$, the link operator can be disentangled into it' left and right parts as:

$$
U = \underbrace{\frac{1}{\sqrt{\hat{N}+1}} \begin{pmatrix} a_2^\dagger(L) & a_1(L) \\ -a_1^\dagger(L) & a_2(L) \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} a_1^\dagger(R) & a_2^\dagger(R) \\ a_2(R) & -a_1(R) \end{pmatrix} \frac{1}{\sqrt{\hat{N}+1}}}_{U_R} \equiv U_L \, U_R \tag{23}
$$

and satisfies $U^\dagger U = U U^\dagger = 1$.

3.3   SU(2) gauge invariant states and Mandelstam constraints

The prepotential operators being associated with sites enable us to construct SU(2) gauge invariant Hilbert spaces at every lattice site. These SU(2) gauge invariant Hilbert spaces at different lattice
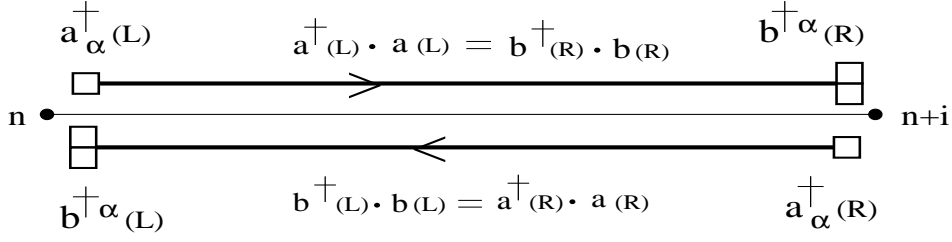
Figure 4: The SU(3) prepotentials and the two $U(1) \otimes U(1)$ oriented abelian flux lines along a link in SU(3) lattice gauge theory. The directions of abelian flux lines are chosen from quark ($a^\dagger$) prepotentials to anti-quark ($b^\dagger$) prepotentials.

sites are mutually orthogonal. Therefore, the Mandelstam constraints which relate the various gauge invariant states, can be analyzed and solved locally at each lattice site. For a d-dimensional lattice we have $2d$ number of prepotential creation operators present at each site all transforming in the same way under the $SU(2)$ group present at the site (see Figure 3). Hence all possible $SU(2)$ invariant creation operators at site n are constructed by anti-symmetrizing any two different prepotential doublets:

$$L_{ij}(n) = \epsilon^{\alpha\beta} a_\alpha^\dagger(n,i) a_\beta^\dagger(n,j) = a^\dagger(n,i) \cdot \tilde{a}^\dagger(n,j), \qquad i,j = 1,2,...,2d \qquad (24)$$

In (24), $a_\alpha^\dagger(n,i)$ with $i = 1,2,\cdots 2d$ denote the 2d prepotentials around the lattice site n (see Figure 3 for $d = 2$). Hence, the most general gauge invariant states at a lattice site n is given by,

$$|\vec{l}(n)\rangle = \prod_{i,j=1}^{2d} (L_{ij}(n))^{l_{ij}(n)} |0\rangle \qquad (25)$$

But these $|\vec{l}(n)\rangle$ states form an over complete basis because of the Mandelstam constraints[4] [8]:

$$(a^\dagger \cdot \tilde{b}^\dagger)(c^\dagger \cdot \tilde{d}^\dagger) \equiv (a^\dagger \cdot \tilde{c}^\dagger)(b^\dagger \cdot \tilde{d}^\dagger) - (a^\dagger \cdot \tilde{d}^\dagger)(b^\dagger \cdot \tilde{c}^\dagger) \qquad (26)$$

A complete orthonormal gauge invariant basis at site n in terms of SU(2) prepotentials is given in terms of SU(2) angular momentum quantum numbers [8]:

$$|LS\rangle_n \equiv |j_1, j_2, ..j_{2d}; j_{12}, j_{123}, ...j_{12..(2d-1)} = j_{2d}\rangle = N(j) \sum_{\{l\}}' \prod_{\substack{i,j \\ i<j}} \frac{1}{l_{ij}!} (L_{ij}(n))^{l_{ij}(n)} |0\rangle \qquad (27)$$

The prime over the summation means that the linking numbers l_{ij} are are summed over all possible values which are consistent with certain geometrical constraints [8]. The states (27) at different lattice sites along with U(1) constraints (18) describe all possible orthonormal (linearly independent) loop states. It is also shown [8] that the loop dynamics for pure SU(2) lattice gauge theory in d dimension is given by real and symmetric $3nj$ Wigner coefficients of the second kind (e.g., n=6, 10 for d=2, 3 respectively).

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[4]We will discuss the Mandelstam constraints in detail in section 4.4.

4   Prepotentials in SU(3) lattice gauge theory

We will now generalize the above SU(2) prepotential formulation to $SU(3)$ lattice gauge theory. Like in SU(2), the SU(3) prepotentials are defined through the left and right electric fields in SU(3) lattice gauge theory. However, now the two fundamental representations 3 (quark) and 3* (anti-quark) of $SU(3)$ are independent. Hence we associate two independent harmonic oscillator prepotential triplets:

$$a_\alpha^\dagger(n, i; L) \equiv a_\alpha^\dagger(L), \qquad b^{\dagger\alpha}(n, i; L) \equiv b^{\dagger\alpha}(L), \qquad \alpha = 1, 2, 3$$

to the left end and

$$a_\alpha^\dagger(n + i, i; R) \equiv a_\alpha^\dagger(R), \qquad b^{\dagger\alpha}(n + i, i; R) \equiv b^{\dagger\alpha}(R), \qquad \alpha = 1, 2, 3$$

to the right end of the link (n, i). Now there are 12 prepotential operators associated with every link. These assignments are shown in Figure 4. Under SU(3) gauge transformation in a d dimensional spatial lattice, the $2d$ $a^\dagger s$ and 2d $b^\dagger s$ on the 2d links emanating from the lattice site n transform as quarks (3) and anti-quarks (3*) respectively. The SU(3) electric fields are:

$$\text{Left electric fields:} \quad E_L^{\mathrm{a}} = \left(a^\dagger(L)\frac{\lambda^{\mathrm{a}}}{2}a(L) - b(L)\frac{\lambda^{\mathrm{a}}}{2}b^\dagger(L) \right)$$

$$\text{Right electric fields:} \quad E_R^{\mathrm{a}} = \left(a^\dagger(R)\frac{\lambda^{\mathrm{a}}}{2}a(R) - b(R)\frac{\lambda^{\mathrm{a}}}{2}b^\dagger(R) \right) \tag{28}$$

In (28), we have used Schwinger boson construction of SU(3) Lie algebra [24, 25]. The electric field generators in (28) generate $SU_L(3) \otimes SU_R(3)$ gauge transformations on every link. The prepotential triplets satisfy the standard harmonic oscillator commutation relations:

$$\left[a^\alpha(l), a_\beta^\dagger(l') \right] = \delta_\beta^\alpha \delta_{l,l'}, \qquad \left[b_\alpha(l), b^{\dagger\beta}(l') \right] = \delta_\alpha^\beta \delta_{l,l'}$$

$$\left[a^\alpha(l), a^\beta(l') \right] = 0, \qquad \left[b_\alpha(l), b_\beta(l') \right] = 0, \quad l, l' = L, R. \tag{29}$$

As all the electric fields in (28) involve both creation and annihilation operators, the number operators in (30) commute with all the electric fields in (28). therefore, the two SU(3) Casimirs on each side of the link (n,i) are:

$$\hat{N}(L) = a^\dagger(L) \cdot a(L), \qquad \hat{N}(R) = a^\dagger(R) \cdot a(R),$$

$$\hat{M}(L) = b^\dagger(L) \cdot b(L), \qquad \hat{M}(R) = b^\dagger(R) \cdot b(R). \tag{30}$$

The eigenvalues of $\hat{N}(L), \hat{M}(L)$ and $\hat{N}(R), \hat{M}(R)$ will be denoted by n_L, m_L and n_R, m_R respectively. We can characterize all the SU(3) irreducible representations on a link by $(n_L, m_L) \otimes (n_R, m_R)$. Using the Gauss law generators (8) and the defining equations (4), the $SU(3)$ gauge transformations of the prepotentials on the left and right side of a link (n, i) are:

$$a_\alpha^\dagger(L) \rightarrow a_\beta^\dagger(L)\left(\Lambda_L^\dagger\right)^\beta{}_\alpha, \qquad a_\alpha^\dagger(R) \rightarrow a_\beta^\dagger(R)\left(\Lambda_R^\dagger\right)^\beta{}_\alpha$$

$$b^{\dagger\alpha}(L) \rightarrow \left(\Lambda_L\right)^\alpha{}_\beta b^{\dagger\beta}(L), \qquad b^{\dagger\alpha}(R) \rightarrow \left(\Lambda_R\right)^\alpha{}_\beta b^{\dagger\beta}(R) \tag{31}$$

The above transformations imply that under SU(3) gauge transformations $a_\alpha^\dagger(L), a_\alpha^\dagger(R)$ transform like quarks and $b^{\dagger\alpha}(L), b^{\dagger\alpha}(R)$ transform like anti-quarks at the left and the right end of the link (n, i) respectively. Therefore, we call $a, a^\dagger$ and $b, b^\dagger$ on various links as quark and anti quark prepotentials respectively.

4.1 The $U(1) \otimes U(1)$ gauge invariance

Like in SU(2) case (see (17)), the defining equations of SU(3) prepotentials (28) are invariant under the following $U(1) \otimes U(1) \otimes U(1) \otimes U(1)$ abelian gauge transformations:

$$a_\alpha^\dagger(L) \to e^{i\theta(L)} a_\alpha^\dagger(L), \qquad\qquad a_\alpha^\dagger(R) \to e^{-i\phi(R)} a_\alpha^\dagger(R),$$
$$b^{\dagger\alpha}(L) \to e^{i\phi(L)} b^{\dagger\alpha}(L), \qquad\qquad b^{\dagger\alpha}(R) \to e^{-i\theta(R)} b^{\dagger\alpha}(R) \qquad (32)$$

In (32), the abelian gauge angles $\theta(l)$ and $\phi(l)$ with $l = L, R$ are defined on the left and right sides of every link. Again like in SU(2) case, the Hilbert space of lattice gauge theory is built by applying the link operators on the vacuum state:

$$U^{\alpha_1}{}_{\beta_1} \; U^{\alpha_2}{}_{\beta_2} \cdots U^{\alpha_n}{}_{\beta_n} |0\rangle$$

and then symmetrizing/anti-symmetrizing αs according to a certain Young tableau. However, this symmetrizing/anti-symmetrizing the left $\alpha \in 3$ indices automatically induces the same symmetries/anti-symmetries on the right $\beta \in 3^*$ indices. This implies that the left and right representations are always conjugate to each other[5], i.e:

$$\hat{N}(L) = \hat{M}(R), \quad\quad \hat{M}(L) = \hat{N}(R). \qquad (33)$$

This implies: $\theta(L) = \theta(R)$ and $\phi(L) = \phi(R)$ on every link. Therefore, besides SU(3) gauge invariance (31) at different lattice sites, the prepotential formulation has additional abelian $U(1) \otimes U(1)$ gauge invariance (32) on every link. The Gauss law constraints (33) imply that abelian fluxes are oriented. We choose the directions of the abelian fluxes on links to be from quark to anti quark prepotentials. To maintain continuity of direction in a loop state the non-abelian fluxes are chosen in the opposite direction (i.e, from anti quark prepotentials to quark prepotentials). These conventions are clearly illustrated on a link in Figure 4 and Figure 5.

4.2 The SU(3) prepotential Hilbert space $\mathcal{H}_\mathbf{p}$

Like in SU(2) case (15), the Hilbert space of SU(3) prepotential operators $\mathcal{H}_p$ can be completely characterized by the following basis on every lattice link:

$$|{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}\rangle_L \otimes |{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}\rangle_R \equiv \hat{L}\,{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \otimes \hat{R}\,{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R. \qquad (34)$$

In (34),

$$\hat{L}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \equiv a_{\alpha_1}^\dagger(L)\cdots a_{\alpha_p}^\dagger(L)b^{\dagger\beta_1}(L)\cdots b^{\dagger\beta_q}(L)|0\rangle_L,$$

and

$$\hat{R}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \equiv a_{\gamma_1}^\dagger(R)\cdots a_{\gamma_q}^\dagger(R)b^{\dagger\delta_1}(R)\cdots b^{\dagger\delta_p}(R)|0\rangle_R$$

are the $SU_L(3) \otimes SU_R(3) \otimes U(1) \otimes U(1)$ flux creation operators on the left and right ends of every link respectively. We have used the $U(1) \otimes U(1)$ Gauss law constraints (33) in (34) with $n_L = m_R = p$ and $m_L = n_R = q$. Note that unlike SU(2) flux creation operators (15) which were SU(2) irreducible, the

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[5]We will analyze the consequences of $E_L^2(n,i) = E_R^2(n+i,i)$ in the next section.

flux operators operators in (34) are SU(3) reducible (see eqns. (38)). In this section we show that this is the reason why, unlike SU(2) case (16), the SU(3) gauge theory Hilbert space $\mathcal{H}_g$ is contained in $\mathcal{H}_p$:

$$\mathcal{H}_g \subset \mathcal{H}_p. \tag{35}$$

Therefore, we now need projection operators to go from $\mathcal{H}_p$ to $\mathcal{H}_g$ (appendix A). This makes SU(3) prepotential analysis slightly more involved than SU(2) (see section 4.3). To appreciate this problem, we start with the following SU(3) gauge invariant state as an example:

$$|\rho_L, \ \rho_R\rangle \equiv \left(a^\dagger(L) \cdot b^\dagger(L)\right)^{\rho_L} \left(a^\dagger(R) \cdot b^\dagger(R)\right)^{\rho_R} |0\rangle. \tag{36}$$

The states (36) are also invariant under $U(1) \otimes U(1)$ gauge transformations (32) if $\rho_L = \rho_R = \rho$ with

Figure 5: The graphical interpretation of the $SU(3) \otimes U(1) \otimes U(1)$ gauge invariant loop state (36) over a link (n, i) with $n_L = n_R = n = 1$. The "magnetic" Sp(2,R) quantum number ρ of this state is non zero ($\rho = 1$) and therefore such states can not be created by the link operators $U(n, i)$. Two types of arrows are used to differentiate abelian and non-abelian fluxes.

$\rho = 0, 1, 2, ..., \infty$. The state (36) with $\rho = 1$ is shown in Figure 5. The gauge invariant states (36) are linear combinations of states in (34):

$$|\rho\rangle \equiv |\rho_L = \rho, \ \rho_R = \rho\rangle = \sum_{\vec{\alpha}} |^{\alpha_1 \alpha_2 \cdots \alpha_\rho}_{\alpha_1 \alpha_2 \cdots \alpha_\rho}\rangle_L \otimes \sum_{\vec{\beta}} |^{\beta_1 \beta_2 \cdots \beta_\rho}_{\beta_1 \beta_2 \cdots \beta_\rho}\rangle_R. \tag{37}$$

However, the only gauge invariant states in pure lattice gauge theories are the Wilson loop states residing around the plaquettes and *not on the links as* $Tr(UU^\dagger) = Tr(U^\dagger U) = 3$ on every link. In other words, the infinite towers of gauge invariant states (36) on different links do not exist in the lattice gauge theory. Infact, this issue of "non gauge theory states" in $\mathcal{H}_p$ is related to the well known multiplicity problem in the direct products of SU(3). Note that the basis (34) in $\mathcal{H}_p$ is obtained by taking two direct products. The states $\hat{L}^{\beta_1 \beta_2 \cdots \beta_q}_{\alpha_1 \alpha_2 \cdots \alpha_p} |0\rangle_L$ and $\hat{R}^{\delta_1 \delta_2 \cdots \delta_p}_{\gamma_1 \gamma_2 \cdots \gamma_q} |0\rangle_R$ are individually direct products of quark and anti-quark irreducible representations: $(n_L = p, 0)_L \otimes (0, m_L = q)_L$ and $(n_R = q, 0)_R \otimes (0, m_R = p)_R$ respectively. Therefore, they can be further reduced using the SU(3) Clebsch Gordan series into irreps. of $SU_L(3)$ and $SU_R(3)$ respectively:

$$(n_L = p, 0)_L \otimes (0, m_L = q)_L \ = \ \sum_{\rho(L)=0}^{min(p,q)} \oplus \underbrace{(p - \rho(L), q - \rho(L))_L}_{\mathcal{H}_p^L(p-\rho(L), q-\rho(L), \rho(L))},$$

$$(n_R = q, 0)_R \otimes (0, m_R = p)_R \ = \ \sum_{\rho(R)=0}^{min(p,q)} \oplus \underbrace{(q - \rho(R), p - \rho(R))_R}_{\mathcal{H}_p^R(q-\rho(R), p-\rho(R), \rho(R))}. \tag{38}$$

The multiplicities[6] occurring in such direct product representations have been extensively studied and classified in [21]. Following [21], we have defined $\mathcal{H}_p^l(p-\rho(l), q-\rho(l), \rho(l)), l = L, R$ mutually orthogonal Hilbert spaces as these Hilbert spaces are in different irreducible representations of $SU_l(3)$. As shown in appendix B, the SU(3) electric field constraints $E_L^2(n, i) = E_R^2(n + i, i)$ along with the $U(1) \otimes U(1)$ Gauss law constraints (33) on links imply:

$$\rho(n, i; L) = \rho(n + i, i; R)$$

in the SU(3) Clebsch Gordan series (38) on every link. Therefore, the prepotential Hilbert space can be classified as:

$$\begin{aligned} \mathcal{H}_p = \prod_{\otimes link} \left\{ \mathcal{H}_p \right\}_{link} &= \prod_{\otimes link} \left\{ \sum_{\rho=0}^{\infty} \sum_{p,q=0}^{\infty} \left(\mathcal{H}_p^L(p, q, \rho) \otimes \mathcal{H}_p^R(q, p, \rho) \right) \right\}_{link} \\ &\equiv \prod_{\otimes link} \left\{ \sum_{\rho=0}^{\infty} \mathcal{H}_p(\rho) \right\}_{link}. \end{aligned} \tag{39}$$

In order to identify the gauge theory Hilbert space $\mathcal{H}_g$ in (39), we define the following three color neutral operators on each side l of every link:

$$k_-(l) \equiv a(l) \cdot b(l), \ k_+(l) \equiv a^\dagger(l) \cdot b^\dagger(l), \ k_0(l) \equiv \frac{1}{2} \left(a^\dagger(l) \cdot a(l) + b^\dagger(l) \cdot b(l) + 3 \right), \quad l \equiv L, R. \tag{40}$$

As usual, we have suppressed the link indices (n, i) in (40). These SU(3) color neutral operators satisfy the Sp(2,R) algebra on both sides of the link:

$$[k_0(l), k_\pm(l')] = \pm\delta_{l,l'} k_\pm(l), \qquad [k_-(l), k_+(l')] = 2\delta_{l,l'} k_0(l), \quad l, l' \equiv L, R. \tag{41}$$

Further, as these Sp(2,R) generators are invariant under SU(3) transformations, they commute with the color electric fields. In other words:

$$[Sp_L(2, R) \otimes Sp_R(2, R), SU_L(3) \otimes SU_R(3)] = 0. \tag{42}$$

Therefore, the Hilbert space of SU(3) lattice gauge theory can be completely and uniquely labeled by $SU_L(3) \otimes Sp_L(2, R) \otimes SU_R(3) \otimes Sp_R(2, R)$ quantum numbers on every link. The irreducible representations of Sp(2,R) are characterized by $|k, \rho\rangle$, where k and ρ represent the Sp(2,R) "spin" and "magnetic" quantum numbers. For the direct product (38) we get [21]: $k(L) = k(R) = \frac{1}{2}(p + q + 3)$. Further $\rho(L) = \rho(R)$ appearing in (38) are the "magnetic quantum numbers" of $Sp_L(2, R) \otimes Sp_R(2, R)$. The raising (lowering) $K_+(K_-)$ operators increase (decrease) the Sp(2,R) magnetic fluxes [21]:

$$|\mathcal{H}_p^L(p, q, \rho \pm 1)\rangle = k_\pm(L) |\mathcal{H}_p^L(p, q, \rho)\rangle, \qquad |\mathcal{H}_p^R(q, p, \rho \pm 1)\rangle = k_\pm(R) |\mathcal{H}_p^R(q, p, \rho)\rangle, \tag{43}$$

where $|\mathcal{H}_p^l(p, q, \rho)\rangle$ denotes an arbitrary vector in $\mathcal{H}_p^l(p, q, \rho)$ with $l = L/R$. In particular, the $\rho = 0$ Hilbert space without any "Sp(2,R) magnetic" flux in (38) is annihilated by k_-:

$$k_-(L) |\mathcal{H}_p^L(p, q, \rho = 0)\rangle = 0, \qquad k_-(R) |\mathcal{H}_p^R(q, p, \rho = 0)\rangle = 0. \tag{44}$$

The equations (43) show that the "spurious gauge invariant" states in (36) are the vectors of one dimensional mutually orthogonal SU(3) invariant Hilbert spaces $\mathcal{H}_p^L(0, 0, \rho) \otimes \mathcal{H}_p^R(0, 0, \rho)$ with $\rho = 1, \cdots \infty$. The strong coupling vacuum is the $\rho = 0$ vacuum.

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[6]Under SU(3) gauge transformations, the vectors in $\mathcal{H}_p^L(p, q, \rho) \otimes \mathcal{H}_p^R(q, p, \rho)$ in (39) transform as $(p, q)_L \otimes (q, p)_R$ irreducible representation of $SU_L(3) \otimes SU_R(3)$ independent of the value of $\rho (= 0, 1, \cdots, \infty)$ leading to infinite multiplicity for each state. While the gauge theory Hilbert space $\mathcal{H}_g$ contains each of these representations only once (see (46)).

4.3 The SU(3) gauge theory Hilbert space $\mathcal{H}_g$

The various flux states in gauge theory Hilbert space $\mathcal{H}_g$ are created by the link matrices $U^\alpha{}_\beta$ acting on the strong coupling vacuum as in (10). Therefore, in order to identify $\mathcal{H}_g$ in $\mathcal{H}_p$ with Sp(2,R) structure (39), we now analyze the Sp(2,R) properties of the link operators in this section. We note that the link matrix $U^\alpha{}_\beta$ can not change the Sp(2,R) magnetic quantum number ρ. As shown at the bottom of Figure 5, $k_+(L) = a^\dagger(L) \cdot b^\dagger(L)$ and $k_+(R) = a^\dagger(R) \cdot b^\dagger(R)$ correspond to three Young tableau boxes in a vertical column (SU(3) singlets) on the left and right side of the links respectively. On the other hand, in terms of the link operators, this left and right anti-symmetrization on a link corresponds to: $\frac{1}{3!}\epsilon_{\alpha_1\alpha_2\alpha_3}\epsilon^{\beta_1\beta_2\beta_3}U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}U^{\alpha_3}{}_{\beta_3} = \det U \equiv 1$ or $tr\,(UU^\dagger) = 3$. Therefore, the states in $\mathcal{H}_g$, obtained by applying link operators on the strong coupling vacuum with $\rho = 0$ ($k_-(l)|0\rangle_l = 0$, $l = L, R$) will also carry $\rho = 0$ quantum numbers. In other words, they too will be annihilated by $k_-(l)$:

$$k_-(L)\Big(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_r}{}_{\beta_r}\Big)|0\rangle = 0, \qquad k_-(R)\Big(U^{\alpha_1}{}_{\beta_1}U^{\alpha_2}{}_{\beta_2}\cdots U^{\alpha_r}{}_{\beta_r}\Big)|0\rangle = 0. \tag{45}$$

Therefore, going back to the classification of $\mathcal{H}_p$ in (39), we identify:

$$\mathcal{H}_g \equiv \prod_{\otimes link} \{\mathcal{H}_p(\rho = 0)\}_{link} \equiv \mathcal{H}_p^0 \tag{46}$$

like in the case of SU(2) lattice gauge theory. In (46) $\mathcal{H}_p^0$ denotes $\rho = 0$ subspace of $\mathcal{H}_p$. Thus the kernel of $(k_-(L)k_-(R))$ in $\mathcal{H}_p$ is the SU(3) gauge theory Hilbert space $\mathcal{H}_g$. Further, (45) implies:

$$[k_-(L), U^\alpha{}_\beta] \simeq 0, \qquad [k_-(R), U^\alpha{}_\beta] \simeq 0, \tag{47}$$

In other words, $k_-(L)$ and $k_-(R)$ weakly commute with the link operators of SU(3) lattice gauge theory[7]. The symbol $\simeq$ in (47) implies that the commutators are zero only when they are applied on the vectors belonging to the gauge theory Hilbert space $\mathcal{H}_g$. We would now like to write the link operators in terms of SU(3) prepotential operators which create SU(3) fluxes only in the gauge theory Hilbert space $\mathcal{H}_g$. This is done in the next section.

4.4 SU(3) irreducible prepotential operators

In this section, we construct the SU(3) irreducible prepotential operators from the prepotential operators in (28) such that they directly create SU(3) irreducible fluxes exactly like in SU(2) case (15). This construction with all the it's group theoretical details is given in [22]. We define the SU(3) irreducible prepotential operators from prepotential operators such that:

1. they have exactly the same $SU(3) \otimes U(1) \otimes U(1)$ quantum numbers,

2. they commute with the Sp(2,R) destruction operator k_-.

As a result, acting on the strong coupling vacuum they directly create the gauge theory Hilbert space $\mathcal{H}_g$ completely bypassing the problem of spurious states like (36) in $\mathcal{H}_p$. we define SU(3) irreducible

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[7]Note that all the electric fields strongly commute with the Sp(2,R) generators (42).

prepotentials [22] as:

$$A_\alpha^\dagger(L) = a_\alpha^\dagger(L) - F_L\, k_+(L)b_\alpha(L), \qquad A_\alpha^\dagger(R) = a_\alpha^\dagger(R) - F_R\, k_+(R)b_\alpha(R),$$
$$B^{\dagger\alpha}(L) = b^{\dagger\alpha}(L) - F_L k_+(L)a^\alpha(L), \qquad B^{\dagger\alpha}(R) = b^{\dagger\alpha}(R) - F_L k_+(R)a^\alpha(R). \qquad (48)$$

In (48), the factors F_L and F_R are given by:

$$F_L = \frac{1}{N(L) + M(L) + 1}, \quad F_R = \frac{1}{N(R) + M(R) + 1}.$$

These factors are chosen so that [22]:

$$\left[k_-(l), A_\alpha^\dagger(l)\right] \simeq 0; \qquad\qquad \left[k_-(l), B^{\dagger\alpha}(l)\right] \simeq 0. \qquad (49)$$

It is easy to check that the irreducible Schwinger boson creation operators commute amongst themselves:

$$\left[A_\alpha^\dagger(l), A_\beta^\dagger(l')\right] = 0, \quad \left[B^{\dagger\alpha}(l), B^{\dagger\beta}(l')\right] = 0, \quad \left[A_\alpha^\dagger(l), B^{\dagger\beta}(l')\right] = 0. \qquad (50)$$

The other commutation relations acting on the SU(3) irreps. are [22]:

$$\left[A^\alpha(l), A_\beta^\dagger(l')\right] \simeq \delta_{ll'}\left(\delta_\alpha^\beta - \frac{1}{N(l) + M(l) + 2}B^{\dagger\alpha}B_\beta\right)$$
$$\left[A^\alpha(l), B^{\dagger\beta}(l')\right] \simeq -\delta_{ll'}\frac{1}{N(l) + M(l) + 2}B^{\dagger\alpha}A^\beta \qquad (51)$$
$$\left[B_\alpha(l), B^{\dagger\beta}(l')\right] \simeq \delta_{ll'}\left(\delta_\beta^\alpha - \frac{1}{N(l) + M(l) + 2}A_\alpha^\dagger A^\beta\right)$$

By construction, $A_\alpha^\dagger(l)$ and $B^{\dagger\alpha}(l)$ transform exactly like $a_\alpha^\dagger(l)$ and $b^{\dagger\alpha}(l)$, $l = L, R$ under $SU(3) \otimes U(1) \otimes U(1)$ and retain the same quantum numbers. Therefore, we can now define:

$$|{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}\rangle_L^0 \otimes |{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}\rangle_R^0 \equiv \hat{\mathcal{L}}\,{}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \otimes \hat{\mathcal{R}}\,{}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R. \qquad (52)$$

In (52), the additional Sp(2,R) quantum numbers $\rho_L = \rho_R = 0$ are put as superscript 0. The operators $\mathcal{L}$ and $\mathcal{R}$ are defined by replacing SU(3) prepotentials in L and R in (34) by the corresponding SU(3) irreducible prepotentials in (48), i.e.,

$$\hat{\mathcal{L}}^{\beta_1\beta_2\cdots\beta_q}_{\alpha_1\alpha_2\cdots\alpha_p}|0\rangle_L \equiv A_{\alpha_1}^\dagger(L)\cdots A_{\alpha_p}^\dagger(L)B^{\dagger\beta_1}(L)\cdots B^{\dagger\beta_q}(L)|0\rangle_L,$$

and

$$\hat{\mathcal{R}}^{\delta_1\delta_2\cdots\delta_p}_{\gamma_1\gamma_2\cdots\gamma_q}|0\rangle_R \equiv A_{\gamma_1}^\dagger(R)\cdots A_{\gamma_q}^\dagger(R)B^{\dagger\delta_1}(R)\cdots B^{\dagger\delta_p}(R)|0\rangle_R.$$

Note that in terms of SU(3) irreducible prepotentials, the "spurious gauge invariant states" like in (36) or (37) do not exist as:

$$A^\dagger(L) \cdot B^\dagger(L)|0\rangle_L \equiv 0, \qquad\qquad A^\dagger(R) \cdot B^\dagger(R)|0\rangle_R \equiv 0. \qquad (53)$$

In other words, the operators $\mathcal{L}$ and $\mathcal{R}$ in (52) are SU(3) irreducible unlike the L and R operators in (34) which are reducible according to (38). In the appendix A we show that $\mathcal{L}$ and $\mathcal{R}$ are related to L and R by projection operators (79). Infact, these SU(3) flux creation operators $\mathcal{L}$ and $\mathcal{R}$ are

$$U{}^{\beta}_{\alpha}\left(\left|\overset{\square\square}{\square}\right\rangle_{(1,1)L}\otimes\left|\overset{\square\square}{\square}\right\rangle_{(1,1)R}\right)=\left(\left|\overset{\square\square\square}{\square}\right\rangle_{(2,1)L}\otimes\left|\overset{\square\square}{\square\square}\right\rangle_{(1,2)R}\right)+\left(\left|\overset{\square}{\square}\right\rangle_{(0,1)L}\otimes\left|\square\right\rangle_{(1,0)R}\right)+\left(\left|\overset{\square\square}{\square\square}\right\rangle_{(0,2)L}\otimes\left|\square\square\right\rangle_{(2,0)R}\right)$$

Figure 6: The Young tableau interpretation of the SU(3) link operator U in terms of the prepotential operators (54) acting on a state with $n_L = m_R \equiv p = 1$ and $m_L = n_R \equiv q = 1$. The three terms in (54) or (55) correspond to the three sets of (mutually conjugate) Young tableaues on the right hand side of this figure respectively. This is SU(3) generalization of Figure 2 for SU(2).

the SU(3) analogues of the SU(2) flux creation operators $\mathcal{L}$ and $\mathcal{R}$ in (15) as both create irreducible fluxes. Further, like in SU(2) case, they bypass the problem of symmetrization and anti symmetrization associated with the link operators. This is because $\hat{\mathcal{L}}$ and $\hat{\mathcal{R}}$ in (52) are defined in terms of SU(3) irreducible prepotential operators which have all the symmetries of SU(3) Young tableaues inbuilt [22]. In other words the role played by SU(2) prepotentials in SU(2) lattice gauge theory is exactly equivalent to the role played by SU(3) irreducible prepotentials in SU(3) lattice gauge theory.

4.5 SU(3) link operators

The SU(3) link operator must create 3 and 3* fluxes at the left and right end of the link and should satisfy the $U(1) \otimes U(1)$ Gauss law constraints (33). These requirements are similar to SU(2) case discussed in section 3.2. The new SU(3) requirement of Sp(2,R) constraint (47) has been solved by defining SU(3) irreducible prepotential operators in the previous section. Noting that by construction, $A^{\dagger}_{\alpha}(l)$ and $B^{\dagger\alpha}(l)$ transform exactly like $a^{\dagger}_{\alpha}(l)$ and $b^{\dagger\alpha}(l)$, $l = L, R$, the general structure of the link operator is:

$$U^{\alpha}{}_{\beta} \;=\; B^{\dagger\alpha}(L)\,\eta\,A^{\dagger}_{\beta}(R) + A^{\alpha}(L)\,\theta\,B_{\beta}(R) + \Big(B(L)\wedge A^{\dagger}(L)\Big)^{\alpha}\,\delta\,\Big(A(R)\wedge B^{\dagger}(R)\Big)_{\beta}. \quad (54)$$

In (54), η, θ and δ are the SU(3) invariants and therefore can only depend on the number operators. These will be fixed later in this section. The link operator constructed in (54) has all the required group theoretical properties:

- Under SU(3) transformations $U(n,i)^{\alpha}{}_{\beta} \rightarrow (\Lambda_L)^{\alpha}{}_{\gamma}U(n,i)^{\gamma}{}_{\delta}(\Lambda_R{}^{\dagger})^{\delta}{}_{\beta}$.

- It is invariant under $U(1) \otimes U(1)$ abelian gauge transformations.

- It creates and destroys fluxes in $\mathcal{H}^0_p$ in (46). It is easy to check that the link operator $U^{\alpha}{}_{\beta}$ in (54) satisfy (47).

- Acting on a link state in $(p,q)_L$ and $(q,p)_R$ representations of $SU(3)_L \times SU(3)_R$:

$$\begin{aligned}U^{\alpha}{}_{\beta}|p,q\rangle_L \otimes |q,p\rangle_R \;=\;& C_1{}^{\alpha}{}_{\beta}|p+1,q\rangle_L \otimes |q,p+1\rangle_R + C_2{}^{\alpha}{}_{\beta}|p,q-1\rangle_L \otimes |q-1,p\rangle_R \\ &+ \; C_3{}^{\alpha}{}_{\beta}|p-1,q+1\rangle_L \otimes |q+1,p-1\rangle_R,\end{aligned} \quad (55)$$

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where C_1, C_2 and C_3 are the SU(3) Clebsch Gordan coefficients. The three terms in (54) correspond to the three terms in (55) respectively. In Figure 6, we illustrate (54) and (55) in terms of SU(3) Young tableau diagrams.

Like in SU(2) case, it is convenient to define left and right link operators as:

$$
U = \underbrace{\begin{pmatrix} B^{\dagger 1}(L)\eta_L & A^1(L)\theta_L & (B(L) \wedge A^\dagger(L))^1 \delta_L \\ B^{\dagger 2}(L)\eta_L & A^2(L)\theta_L & (B(L) \wedge A^\dagger(L))^2 \delta_L \\ B^{\dagger 3}(L)\eta_L & A^3(L)\theta_L & (B(L) \wedge A^\dagger(L))^3 \delta_L \end{pmatrix}}_{U_L} \underbrace{\begin{pmatrix} A^1(R)\bar\eta_R & B^{1\dagger}(R)\bar\theta_R & (B(R) \wedge A^\dagger(R))^1 \bar\delta_R \\ A^2(R)\bar\eta_R & B^{2\dagger}(R)\bar\theta_R & (B(R) \wedge A^\dagger(R))^2 \bar\delta_R \\ A^3(R)\bar\eta_R & B^{3\dagger}(R)\bar\theta_R & (B(R) \wedge A^\dagger(R))^3 \bar\delta_R \end{pmatrix}^\dagger}_{U_R} \quad (56)
$$

Where $\eta_L, \theta_L, \delta_L$ and $\bar\eta_R, \bar\theta_R, \bar\delta_R$ are the left and right invariants constructed out of the number operators. From (54):

$$
\eta = \eta_L \; \eta_R, \qquad \theta = \theta_L \; \theta_R, \qquad \delta = \delta_L \; \delta_R. \quad (57)
$$

From (56):

$$
U_L^\dagger U_L = \begin{pmatrix} \bar\eta_L \left(B \cdot B^\dagger \right) \eta_L & \bar\eta_L \underbrace{\left(B \cdot A \right)}_{\simeq 0} \theta_L & \bar\eta_L \underbrace{\left(B \cdot (B \wedge A^\dagger) \right)}_{\equiv 0} \delta_L \\ \bar\theta_L \underbrace{\left(A^\dagger \cdot B^\dagger \right)}_{\simeq 0} \eta_L & \bar\theta_L \left(A^\dagger \cdot A \right) \theta_L & \bar\theta_L \underbrace{\left(A^\dagger \cdot (B \wedge A^\dagger) \right)}_{\equiv 0} \delta_L \\ \bar\delta_L \underbrace{\left((B^\dagger \wedge A) \cdot B^\dagger \right)}_{\equiv 0} \eta_L & \bar\delta_L \underbrace{\left((B^\dagger \wedge A) \cdot A \right)}_{\equiv 0} \theta_L & \bar\delta_L \left((A \wedge B^\dagger) \cdot (B \wedge A^\dagger) \right) \delta_L \end{pmatrix} \quad (58)
$$

Similarly,

$$
U_R U_R^\dagger = \begin{pmatrix} \eta_R \left(A^\dagger \cdot A \right) \bar\eta_R & \eta_R \underbrace{\left(A^\dagger \cdot B^\dagger \right)}_{\simeq 0} \bar\theta_R & \eta_R \underbrace{\left(A^\dagger \cdot (B \wedge A^\dagger) \right)}_{\equiv 0} \bar\delta_R \\ \theta_R \underbrace{\left(B \cdot A \right)}_{\simeq 0} \bar\eta_R & \theta_R \left(B \cdot B^\dagger \right) \bar\theta_R & \theta_R \underbrace{\left(B \cdot (B \wedge A^\dagger) \right)}_{\equiv 0} \bar\delta_R \\ \delta_R \underbrace{\left((B^\dagger \wedge A) \cdot A \right)}_{\equiv 0} \bar\eta_R & \delta_R \underbrace{\left((B^\dagger \wedge A) \cdot B^\dagger \right)}_{\equiv 0} \bar\theta_R & \delta_R \left((A \wedge B^\dagger) \cdot (B \wedge A^\dagger) \right) \bar\delta_R \end{pmatrix} \quad (59)
$$

In (58) and (59), we have suppressed the L/R indices from the prepotential operators $(A, A^\dagger)$ and $(B, B^\dagger)$. Demanding $U_L^\dagger U_L = 1$ and $U_R U_R^\dagger = 1$, we get:

$$
\eta_L = \frac{1}{\sqrt{B(L) \cdot B^\dagger(L)}}, \quad \theta_L = \frac{1}{\sqrt{A^\dagger(L) \cdot A(L)}}, \quad \delta_L = \frac{1}{\sqrt{(A(L) \wedge B^\dagger(L)) \cdot (B(L) \wedge A^\dagger(L))}};
$$

$$
\eta_R = \frac{1}{\sqrt{A^\dagger(R) \cdot A(R)}}, \quad \theta_R = \frac{1}{\sqrt{B(R) \cdot B^\dagger(R)}}, \quad \delta_R = \frac{1}{\sqrt{(A(R) \wedge B^\dagger(R)) \cdot (B(R) \wedge A^\dagger(R))}}. \quad (60)
$$

The link operators in (54) with (57) and (60) satisfy: $UU^\dagger = U^\dagger U = 1$. Having written the link operators in terms of the SU(3) irreducible prepotentials, we now cast the left and right electric fields (28) in terms of $A(l), A^\dagger(l), B(l), B^\dagger(l)$ with $l = L, R$. Using the very special structures of the SU(3) irreducible prepotentials in (48), it is easy to check that:

$$
\begin{aligned}
E_L^a &= \left(a^\dagger(L)\frac{\lambda^a}{2}a(L) - b(L)\frac{\lambda^a}{2}b^\dagger(L) \right) \simeq \left(A^\dagger(L)\frac{\lambda^a}{2}A(L) - B(L)\frac{\lambda^a}{2}B^\dagger(L) \right) \\
E_R^a &= \left(a^\dagger(R)\frac{\lambda^a}{2}a(R) - b(R)\frac{\lambda^a}{2}b^\dagger(R) \right) \simeq \left(A^\dagger(R)\frac{\lambda^a}{2}A(R) - B(R)\frac{\lambda^a}{2}B^\dagger(R) \right)
\end{aligned} \quad (61)
$$

In (61), we have made use of the identities: $a(L) \cdot b(L) \equiv k_-(L) \simeq 0$ and $a(R) \cdot b(R) \equiv k_-(R) \simeq 0$ on every link of the lattice. In fact the results (61) were expected because $(a_\alpha^\dagger, b^{\dagger\beta})$ and $(A_\alpha^\dagger, B^{\dagger\beta})$ have exactly the same $SU(3) \otimes U(1) \otimes U(1)$ transformation properties.
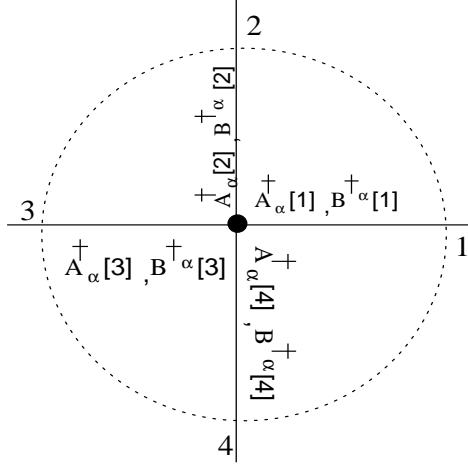

Figure 7: The SU(3) prepotentials associated with a lattice site n in $d = 2$. This is SU(3) generalization of Figure (3) for SU(2).

4.6   SU(3) gauge invariant states and Mandelstam Constraints

In this section we construct all possible SU(3) gauge invariant states at a given lattice site using prepotential approach. We also discuss the Mandelstam constraints which relate these gauge invariant states. The additional $U(1) \otimes U(1)$ Gauss law (33) can be satisfied by drawing the abelian flux lines along the links as is done in Figure 8. As shown in Figure 7, every lattice site in 2d space dimension is associated with $2d$ pairs of quark-anti quark prepotentials $(A_\alpha^\dagger, B^{\dagger\alpha})$. Under a gauge transformation at site n, all these $2d$ quark (anti quark) prepotentials transform together as triplet (anti-triplet). Therefore, the fundamental SU(3) gauge invariant creation operator vertices at a lattice site n are:

$$
\begin{align}
L_{[ij]} &\equiv A^\dagger[i] \cdot B^\dagger[j], \qquad i \neq j, \tag{62} \\
A_{[i_1, i_2, i_3]} &= \epsilon^{\alpha_1 \alpha_2 \alpha_3} A_{\alpha_1}^\dagger[i_1] A_{\alpha_2}^\dagger[i_2] A_{\alpha_3}^\dagger[i_3], \tag{63} \\
B_{[j_1, j_2, j_3]} &= \epsilon_{\beta_1 \beta_2 \beta_3} B^{\dagger\beta_1}[j_1] B^{\dagger\beta_2}[j_2] B^{\dagger\beta_3}[j_3] \tag{64}
\end{align}
$$

These vertices are shown in Figure 8. We have taken $i \neq j$ in (62) because $L_{ii} = A^\dagger[i] \cdot B^\dagger[i] \simeq 0$, $i, j = 1, 2, \cdots 2d$ according to (47). Also, $A_{[i_1, i_2, i_3]}$ and $B_{[j_1, j_2, j_3]}$ are completely anti-symmetric in (i_1, i_2, i_3) and (j_1, j_2, j_3) indices respectively. The above

$$
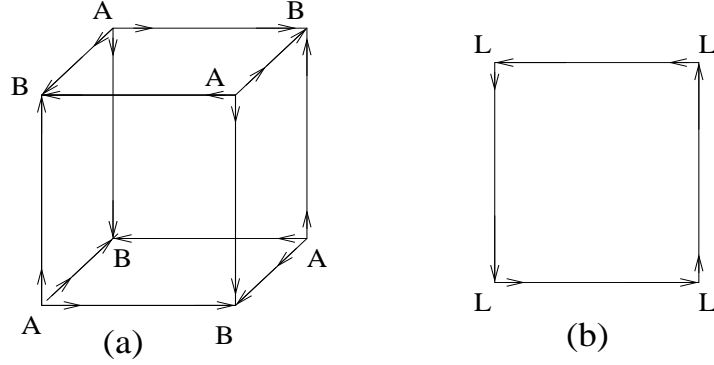2(^{2d}C_2) + 2(^{2d}C_3) = \frac{2d(2d-1)(2d+1)}{3}
$$

Figure 8: Graphical representation of the three possible SU(3) gauge invariant L, A, B types of vertices. Two simple $SU(3) \otimes U(1) \otimes U(1)$ gauge invariant loop states are also shown. The arrows represent the directions of the abelian (non-abelian) fluxes on the links (sites).

basic SU(3) gauge invariant operators enable us to write the most general SU(3) gauge invariant state at a given lattice site as:

$$|\vec{l}_{[ij]}, \vec{p}_{[i_1 i_2 i_3]}, \vec{q}_{[j_1 j_2 j_3]}\rangle = \prod_{\substack{i,j=1 \\ i \neq j}}^{2d} \left(L_{[ij]}\right)^{l_{[ij]}} \prod_{[i_1 i_2 i_3]=1}^{2d C_3} \left(A_{[i_1 i_2 i_3]}\right)^{p_{[i_1 i_2 i_3]}} \prod_{[j_1 j_2 j_3]=1}^{2d C_3} \left(B_{[j_1 j_2 j_3]}\right)^{q_{[j_j i_2 j_3]}} |0\rangle. \quad (65)$$

In (65), $\vec{l}_{[ij]}, \vec{p}_{[i_1 i_2 i_3]}, \vec{q}_{[j_1 j_2 j_3]}$ are $\frac{2d(2d-1)(2d+1)}{3}$ non-negative integers describing all possible SU(3) gauge invariant states at a given lattice site. The various possible loop states set in pure SU(3) lattice gauge theory are direct products of (65) at various lattice sites consistent with $U(1) \otimes U(1)$ Gauss law (33) along every link.

As in the loop formulation where various loop states are mutually related by Mandelstam constraints, not all states in (65) are linearly independent. Infact, in the present SU(3) prepotential formulation (like in SU(2) case) the Mandelstam constraints become local and take very simple forms in terms of the SU(3) gauge invariant vertices in (62,63) and (64) at every lattice site n. We start with the simplest SU(3) Mandelstam constraints:

$$A_{[i_1, i_2, i_3]} B_{[j_1, j_2, j_3]} \equiv \sum_{\{s_1, s_2, s_3\} \in S_3} (-1)^s L_{[i_1 j_{s_1}]} L_{[i_2 j_{s_2}]} L_{[i_3 j_{s_3}]}. \quad (66)$$

In (66), S_3 denotes the permutation group of order 3, $\{s_1, s_2, s_3\}$ denote the 3! permutations of $\{1, 2, 3\}$ and s is the parity of permutation. In other words, the Mandelstam constraints (66) state that the A and B type vertices annihilate each other in pairs to produce L type vertices. The constraints (66) are illustrated in Figure 9. Therefore, the SU(3) gauge invariant states of $(L - A - B)$ type in (65) can always be written either as $(L - A)$ type or as $(L - B)$ type at each lattice site. It is interesting to analyze the Mandelstam constraints discussed in [4] in terms of SU(3) prepotential operators. Following [4], we consider the set of r $(r > 3)$ loops $C_1(n), C_2(n), \cdots, C_r(n)$ all based at lattice site n. These loops start from n in the direction $i_1, i_2, \cdots i_r$ and come back to n from directions $j_1, j_2, \cdots j_r$ respectively.
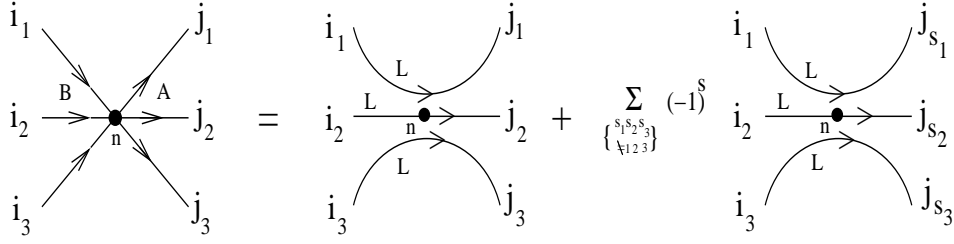
19

Figure 9: The graphical representation of local SU(3) Mandelstam constraints (66) in terms of SU(3) gauge invariant vertices A, B and L constructed out of the SU(3) irreducible prepotential operators at a lattice site n. The A and B type of vertices at n annihilate each other to produce L type of vertices.

Then the products of these Wilson loops satisfies:

$$\sum_{\substack{\alpha_{i_1}\cdots\alpha_{i_r}\\ \beta_{j_1}\cdots\beta_{j_r}}} \epsilon_{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}} \epsilon^{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}} \left(W(C_1(n))\right)^{\alpha_{j_1}}{}_{\beta_{i_1}} \left(W(C_2(n))\right)^{\alpha_{j_2}}{}_{\beta_{i_2}} \cdots \left(W(C_r(n))\right)^{\alpha_{j_r}}{}_{\beta_{i_r}} \equiv 0. \qquad (67)$$

Using the identities

$$\epsilon_{\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_r}} \epsilon^{\beta_{j_1}\beta_{j_2}\cdots\beta_{j_r}} = \delta^{\beta_{j_1}}_{\alpha_{i_1}} \delta^{\beta_{j_2}}_{\alpha_{i_2}} \cdots \delta^{\beta_{j_r}}_{\alpha_{i_r}} - \delta^{\beta_{j_1}}_{\alpha_{i_2}} \delta^{\beta_{j_2}}_{\alpha_{i_i}} \cdots \delta^{\beta_{j_r}}_{\alpha_{i_r}} + \cdots$$

(68) can be written in terms of traces of Wilson loops [4]:

$$TrW(C_1)TrW(C_2)\cdots TrW(C_r) - TrW(C_1 C_2)TrW(C_3)TrW(C_4)\cdots TrW(C_r) + \cdots = 0. \qquad (68)$$

The Mandelstam constraints (68) in terms of the link operators represent highly non-local constraints as one can always choose the loops $C_1, C_2, \cdots C_r$ to be as large as one wishes. *However, in terms of the prepotentials the constraints (68) become local.* All one has to do is to replace the Wilson loops in (68) by the prepotentials which are attached to their starting and end points, i.e,:

$$W(C_s)^{\alpha_{i_s}}{}_{\beta_{i_s}} \rightarrow L^{\alpha_{j_s}}{}_{\beta_{i_s}} \equiv B^{\dagger \alpha_{j_s}} A^{\dagger}_{\beta_{i_s}}, \quad s = 1, 2, \cdots, r. \qquad (69)$$

Note that unlike non-local Wilson loop $W(C_s)$, the operators $B^{\alpha_{j_s}}$ and $A^{\dagger}_{\beta_{i_s}}$ and hence $L^{\alpha_{j_s}}{}_{\beta_{i_s}}$ are completely defined at lattice site n. Noting that $Tr\left(L^{\alpha_{j_s}}{}_{\beta_{i_s}}\right) = L[i_s j_s]$, the non-local Mandelstam constraints (68) acquire the following simple local form:

$$\sum_{\{s_1, s_2, \cdots s_r\} \in S_r} (-1)^s L_{[i_1 j_{s_1}]} L_{[i_2 j_{s_2}]} \cdots L_{[i_3 j_{s_r}]} = 0 \qquad (70)$$

and are illustrated in Figure 10. *Note that all the unnecessary details like shapes, sizes and lengths of the loops $C_1, C_2, \cdots C_r$ in (68) disappear in the corresponding prepotential form (70).*

4.6.1 The solutions

The Mandelstam constraints in their present local prepotential forms (66) and (70), instead of non-local form (68) in terms of link operators, are now accessible to explicit local solutions like in SU(2) lattice gauge theory [8]. Note that they are still infinite in number at every lattice site. The solutions

Figure 10: The graphical representation of local SU(3) Mandelstam constraints (70) involving only L type of vertices at lattice site n.

must be all possible mutually independent linear combinations of the states in (65) at a given lattice site. Following the techniques discussed in [10] in the context of duality transformations in lattice gauge theories, these linear combinations can be obtained by characterizing the resultant states at a site n by their complete SU(3) quantum numbers with the net SU(3) fluxes being zero. This will be SU(3) analogue of SU(2) result (27). The quantum numbers needed to specify such states can be easily computed [10] as follows. In d dimension, there are 2d links emanating from a lattice site n. Each of these 2d directions is attached with SU(3) operators $(A^{\dagger}[i], B^{\dagger}[i], i = 1, 2, \cdots, 2d)$ as shown in Figure 7. Therefore, there are 2d Hilbert spaces associated with a lattice site and each can be characterized by it's SU(3) quantum numbers. In the standard language [24], the SU(3) irreducible representations are completely specified by 5 quantum numbers: $|p, q, i^2, i_z, y\rangle$ where p and q are the eigenvalues of two SU(3) Casimir operators and i, i_z, y are the SU(3) "magnetic" quantum numbers representing SU(2) spin, it's third component and hyper charge respectively. In the present language with constraints, each of the 2d directions (see Figure 7) is associated with 6 harmonic oscillators $(A^{\dagger}[i], B^{\dagger}[i], i = 1, 2, \cdots, 2d)$ and therefore requires 6 occupation numbers to completely specify the basis. The constraints $k_{-}[i] \equiv a[i] \cdot b[i] \simeq 0$ reduces this to 5 in each direction. Therefore $5 \times 2d = 10d$ quantum numbers are needed to specify a local Hilbert space basis completely at each lattice site. Not all these quantum numbers are independent as 2d of these are related to the previous sites by $U(1) \otimes U(1)$ Gauss law constraints (33). Therefore, we are left with 8d quantum numbers at every lattice site. Finally, the SU(3) gauge invariance further implies 8 constraints. Therefore, the net independent quantum numbers are $8(d-1)$ per lattice site. As expected, this is the the number of transverse degree of freedom of 8 SU(3) gluons in d dimension at every lattice site. The abelian $U(1) \times U(1)$ fluxes over the links will now glue these local SU(3) invariant orthogonal basis at neighboring lattice sites according to their Gauss laws (33). This will give complete solutions of all the SU(3) Mandelstam constraints like what what was done in SU(2) lattice gauge theory [8]. Infact, the addition of fluxes in SU(3) lattice gauge theory has been discussed in [27]). These results combined with the results of this work should enable us to solve SU(3) Mandelstam constraints completely in terms of vertex operators of section (4.6). This explicit construction of all the independent SU(3) loop states and their dynamics along the line of [8] is in progress and will be reported elsewhere.

5 Summary and discussion

In this work we analyze SU(3) lattice gauge theory in terms of the prepotential operators which under gauge transformations transform like fundamental matter fields. We constructed the SU(3) irreducible prepotential operators which acting on strong coupling vacuum directly created the QCD fluxes around lattice sites. All SU(3) gauge invariant vertices in terms of these QCD flux operators were constructed at every lattice site. These SU(3) invariant vertices, in turn, enabled us to cast all SU(3) Mandelstam constraints in their local forms. As mentioned in the text this is an essential step towards their complete solution. The complete solution of Mandelstam constraints, in turn, will allow us to write down SU(3) lattice gauge theory completely and exactly in terms of minimum essential gauge invariant loop and string co-ordinates without any redundant loop/strings degrees of freedom. The prepotential operators also allow us to simplify lattice gauge theory Hamiltonian as given in (1). In particular, for the present SU(3) case, one can simply replace the plaquette or magnetic term $TrU_{plaquette}$ in (1) by a new plaquette interaction consisting of the 4 L type vertices at the 4 corners of every plaquette. Note that the new Hamiltonian constructed this was has exactly the same symmetries as (1) and therefore expected to be in the same universality class. The addition of matter field interactions in the prepotential formulation is trivial as matter and prepotential have similar SU(3) gauge transformation properties. The difference lies in the abelian $U(1) \otimes U(1)$ transformations under which matter fields remain invariant.

The results in this work can also be generalized to SU(N) lattice gauge theory. We can use SU(N) Schwinger bosons [26] or prepotentials to construct SU(N) electric fields on lattice similar to (11) and (28). We need to elevate these prepotentials so that they have symmetries of SU(N) Young tableaues inbuilt. As in section (4.6), the SU(N) Mandelstam constraints will again be local and can be solved using the techniques discussed in this work. The work in this direction is in progress and will be reported elsewhere.

A   The projection operators in $\mathcal{H}_p$:

In this appendix we briefly discuss the construction of projection operators which project $\mathcal{H}_p$ to $\mathcal{H}_g$ on every link:

$$\mathcal{P} \ \{\mathcal{H}_p\}_{link} = \{\mathcal{H}_p(\rho = 0)\}_{link} = \{\mathcal{H}_g\}_{link}. \tag{71}$$

The group theoretical details of this construction can be found in [22]. It is convenient to first break up $\{\mathcal{H}_p\}_{link}$ into Hilbert spaces containing p (q) quarks and q (p) anti-quark prepotentials on the left (right):

$$\{\mathcal{H}_p\}_{link} = \sum_{p,q=0}^{\infty} \oplus \{\mathcal{H}_p\}_{link} (p,q). \tag{72}$$

These subspaces $\{\mathcal{H}_p\}_{link}(p,q)$ are themselves direct products of left and right Hilbert spaces:

$$\{\mathcal{H}_p\}_{link}(p,q) = \{\mathcal{H}_p^L\}_{link}(p,q) \otimes \{\mathcal{H}_p^R\}_{link}(q,p). \tag{73}$$

The basis vectors spanning $\{\mathcal{H}_p^l\}_{link}(p,q), l = L, R$ are given in terms of left and right flux creation operators:

$$\hat{L}_{\alpha_1\alpha_2\cdots\alpha_p}^{\beta_1\beta_2\cdots\beta_q}|0\rangle_L \equiv a_{\alpha_1}^\dagger(L)\cdots a_{\alpha_p}^\dagger(L)b^{\dagger\beta_1}(L)\cdots b^{\dagger\beta_q}(L)|0\rangle_L,$$

and

$$\hat{R}_{\gamma_1\gamma_2\cdots\gamma_q}^{\delta_1\delta_2\cdots\delta_p}|0\rangle_R \equiv a_{\gamma_1}^\dagger(R)\cdots a_{\gamma_q}^\dagger(R)b^{\dagger\delta_1}(R)\cdots b^{\dagger\delta_p}(R)|0\rangle_R$$

in (34). We now construct the projection operators $\mathcal{P}_l(p,q)$ in each of these subspaces with

$$\mathcal{P} = \sum_{p,q=0}^{\infty} \oplus \mathcal{P}_L(p,q) \otimes \mathcal{P}_R(q,p). \tag{74}$$

The left and right projection operators $\mathcal{P}_l(p,q), l = L, R$ are of the form [22]:

$$\mathcal{P}_L(p,q) \equiv \sum_{r=0}^{\infty} g_r(p,q)\left(k_+(L)\right)^r\left(k_-(L)\right)^r$$
$$\mathcal{P}_R(q,p) \equiv \sum_{r=0}^{\infty} h_r(q,p)\left(k_+(R)\right)^r\left(k_-(R)\right)^r \tag{75}$$

The unknown coefficients g and h in (75) are fixed by demanding the Sp(2,R) constraints (44):

$$k_-(L)\mathcal{P}_L(p,q)\{\mathcal{H}_p^L\}_{link}(p,q) = 0, \quad k_-(R)\mathcal{P}_R(q,p)\{\mathcal{H}_p^R\}_{link}(q,p) = 0. \tag{76}$$

The solutions of the equations (76) are [22]:

$$g_r(p,q) = h_r(q,p) = \frac{(-1)^r}{r!}\frac{(p+q+1-r)!}{(p+q+1)!}, \tag{77}$$

leading to:

$$\mathcal{P}_L(p,q) = \frac{1}{(p+q+1)!}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(p+q+1-r)!\left(k_+(L)\right)^r\left(k_-(L)\right)^r,$$
$$\mathcal{P}_R(q,p) = \frac{1}{(p+q+1)!}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!}(p+q+1-r)!\left(k_+(R)\right)^r\left(k_-(R)\right)^r. \tag{78}$$

Note that the SU(3) irreducible prepotentials in (48) already commute with the Sp(2,R) constraints (49) and therefore acting on the strong coupling vacuum directly generate the gauge theory Hilbert space $\mathcal{H}_g$. In other words:

$$\hat{\mathcal{L}}\,_{\alpha_1\alpha_2\cdots\alpha_p}^{\beta_1\beta_2\cdots\beta_q}|0\rangle_L = \mathcal{P}_L(p,q)\hat{L}\,_{\alpha_1\alpha_2\cdots\alpha_p}^{\beta_1\beta_2\cdots\beta_q}|0\rangle_L, \quad \hat{\mathcal{R}}\,_{\gamma_1\gamma_2\cdots\gamma_q}^{\delta_1\delta_2\cdots\delta_p}|0\rangle_R = \mathcal{P}_R(q,p)\hat{R}\,_{\gamma_1\gamma_2\cdots\gamma_q}^{\delta_1\delta_2\cdots\delta_p}|0\rangle_R \tag{79}$$

are the relations amongst the SU(3) reducible and irreducible flux operators on the left and right side of every link.

B   The electric field constraints

Using the λ matrix identity: $\sum_{a=1}^{8} \left(\frac{\lambda^a}{2}\right)_\beta^\alpha \left(\frac{\lambda^a}{2}\right)_\sigma^\gamma = \frac{1}{2}\delta_\sigma^\alpha \delta_\beta^\gamma - \frac{1}{6}\delta_\beta^\alpha \delta_\sigma^\gamma$, the squares of left and right electric fields can be written as:

$$\sum_{a=1}^{8} E_L^a(n,i)E_L^a(n,i) = \hat{N}(L)\left(\frac{\hat{N}(L)}{3}+1\right) + \hat{M}(L)\left(\frac{\hat{M}(L)}{3}+1\right) - k_+(L)k_-(L) + \frac{1}{3}\hat{N}(L)\hat{M}(L)$$

$$\sum_{a=1}^{8} E_R^a(n,i)E_R^a(n,i) = \hat{N}(R)\left(\frac{\hat{N}(R)}{3}+1\right) + \hat{M}(R)\left(\frac{\hat{M}(R)}{3}+1\right) - k_+(R)k_-(R) + \frac{1}{3}\hat{N}(R)\hat{M}(R).$$

The electric field constraints (6) along with the $U(1) \otimes U(1)$ Gauss law constraints (33) imply:

$$k_+(L)k_-(L) = k_+(R)k_-(R). \tag{80}$$

On the other hand, the action of k_+k_- on a general Sp(2,R) irrep. $|k,m\rangle$ is given by [21]:

$$k_+k_-|k,m\rangle = (m-k)(m+k-1)|k,m\rangle, \tag{81}$$

where, $m = k + \rho$. In the present case the electric field constraint (80) and the eigenvalue equation (81) imply:

$$(m(L)-k(L))\,(m(L)+k(L)-1) = (m(R)-k(R))\,(m(R)+k(R)-1)\,. \tag{82}$$

As $k(L) = k(R) = \frac{1}{2}(p+q+3)$, we get the unique solution of (82):

$$\rho_L(n,i) = \rho_R(n+i,i). \tag{83}$$

Therefore, in the prepotential Hilbert space $\mathcal{H}_p$ the left and the right Sp(2,R) "magnetic" quantum numbers are same on every link.

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